

Real splitting and stability index for algebras with involution

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Abstract

For central simple algebras with involution over a formally real field, “splitting” by real closures and maximal ordered subfields is investigated, and another presentation of H -signatures of hermitian forms is given. The stability index for central simple algebras with involution is defined and studied.

Key words. Central simple algebra, involution, formally real field, splitting, hermitian form, H -signature, stability index

2010 MSC. 16K20, 11E39, 13J30

1 Introduction

It is well-known that any central simple algebra A possesses a splitting field, i.e. a field extension L of its centre Z such that $A \otimes_Z L$ is Brauer equivalent to L . For example, any algebraic closure of Z is a splitting field of A . By a theorem of Wedderburn, A is isomorphic to a matrix algebra over a central division Z -algebra D . Any maximal subfield of D is a splitting field of D and thus a splitting field of A .

In this paper we study the “real splitting” of an algebra with involution, motivated by the theory of H -signatures, developed in [2] and [1]. More precisely: let us assume that A is a simple F -algebra, equipped with an F -linear involution σ , where F is a formally real field. Let ϑ be an involution on D of the same kind as σ . Let P be an ordering of F and let F_P be a real closure of F at P . By a theorem of Frobenius, $A \otimes_F F_P$ is Brauer equivalent to one of F_P , $F_P(\sqrt{-1})$ or $(-1, -1)_{F_P}$ whenever $A \otimes_F F_P$ is simple. In analogy with the first paragraph, it is a natural question to ask if there exists a maximal ordered extension (L, Q) of (F, P) , contained in D , which behaves like F_P in the sense that $D \otimes_F L$ (and thus $A \otimes_F L$) is Brauer equivalent to one of L , $L(\sqrt{-d})$ or $(-a, -b)_L$ with $a, b, d \in Q$ whenever $D \otimes_F L$ is simple. We answer this question in the affirmative and in addition obtain

more precise results about the nature of the field L , which take into account the involution ϑ .

We use this result to give another presentation of the theory of H -signatures and also introduce and study the stability index of algebras with involution.

2 Notation

We give a brief overview of the notation used in this paper and refer to the standard references [8], [9], [10] and [16] as well as [2] and [1] for the details.

For a ring A and an involution σ on A , we denote the set of symmetric elements of A with respect to σ by $\text{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$.

Let F be a formally real field with space of orderings X_F and Witt ring $W(F)$. For an ordering $P \in X_F$ we denote by F_P a real closure of F at P . By an F -algebra with involution we mean a pair (A, σ) where A is a finite-dimensional F -algebra with centre $Z(A)$, equipped with an involution $\sigma : A \rightarrow A$, such that $F = Z(A) \cap \text{Sym}(A, \sigma)$ and which is assumed to be either simple (if $Z(A)$ is a field) or a direct product of two simple algebras (if $Z(A) = F \times F$). Observe that $\dim_F Z(A) =: \kappa \leq 2$. If A is a division algebra, we call (A, σ) an F -division algebra with involution. By [9, Prop. 2.14] we may assume that σ is the exchange involution when $Z(A) = F \times F$.

When $\kappa = 1$, we say that σ is *of the first kind*. When $\kappa = 2$, we say that σ is *of the second kind* (or of *unitary type*). Note that $\sigma|_{Z(A)}$ is the non-trivial F -automorphism of $Z(A)$ in this case. Assume $\kappa = 1$ and $\dim_F A = m^2 \in \mathbb{N}$. Then σ is either of *orthogonal type* (if $\dim_F \text{Sym}(A, \sigma) = m(m+1)/2$) or of *symplectic type* (if $\dim_F \text{Sym}(A, \sigma) = m(m-1)/2$).

It follows from the structure theory of F -algebras with involution that A is isomorphic to a full matrix algebra $M_n(D)$ for a unique $n \in \mathbb{N}$ and an F -division algebra D (unique up to isomorphism) which is equipped with an involution ϑ of the same kind as σ , cf. [9, Thm. 3.1]. We denote Brauer equivalence by \sim .

Let (A, σ) and (B, τ) be F -algebras with involution of the same kind. If A and B are Brauer equivalent, then (A, σ) and (B, τ) are Morita equivalent, cf. [5, Ex. 1.4].

For $\varepsilon \in \{-1, 1\}$ we write $W_\varepsilon(A, \sigma)$ for the *Witt group* of Witt equivalence classes of ε -hermitian forms $h : M \times M \rightarrow A$, defined on finitely generated right A -modules M . All forms in this paper are assumed to be non-singular and are identified with their Witt equivalence classes. We write $+$ for the sum in both $W(F)$ and $W_\varepsilon(A, \sigma)$. The group $W_\varepsilon(A, \sigma)$ is a $W(F)$ -module and we denote the product of $q \in W(F)$ and $h \in W_\varepsilon(A, \sigma)$ by $q \cdot h$.

Assume that (A, σ) and (B, τ) are Morita equivalent F -algebras with involution of the same kind. It follows from [8, Thm. 9.3.5] (for full details, see [6, Chap. 2])

that there exists an isomorphism $W_\varepsilon(A, \sigma) \simeq W_{\varepsilon_0}(B, \tau)$, where $\varepsilon_0 = 1$ if σ and τ are both orthogonal or both symplectic, and $\varepsilon_0 = -1$ otherwise. If σ and τ are both unitary, then the isomorphism holds for any $\varepsilon_0 \in \{-1, 1\}$, cf. [2, Lemma 2.1(iii)].

In the context of signatures later on (Section 4), we will consider non-trivial morphisms from $W(A \otimes_F F_P, \sigma \otimes \text{id})$ to \mathbb{Z} and therefore need to know when $W(A \otimes_F F_P, \sigma \otimes \text{id})$ is torsion, which motivates the following definition.

Definition 2.1. Let (A, σ) be an F -algebra with involution. We define the set of *nil-orderings* of (A, σ) as follows

$$\text{Nil}[A, \sigma] := \{P \in X_F \mid W(A \otimes_F F_P, \sigma \otimes \text{id}) \text{ is torsion}\}.$$

For convenience we also introduce

$$\widetilde{X}_F := X_F \setminus \text{Nil}[A, \sigma],$$

which does not indicate the dependency on (A, σ) in order to avoid cumbersome notation.

Let R be a real closed field and let (A, σ) be an R -algebra with involution. It follows from well-known theorems of Wedderburn and Frobenius and an application of Morita theory that $W(A, \sigma)$ is isomorphic to one of the following Witt groups:

$$\begin{aligned} W(R, \text{id}) &\simeq W_{\pm 1}(R(\sqrt{-1}), -) \simeq W((-1, -1)_R, -) \simeq \mathbb{Z} \\ W_{-1}(R, \text{id}) &\simeq W_{\pm 1}(R \times R, \widehat{}) \simeq W_{\pm 1}((-1, -1)_R \times (-1, -1)_R, \widehat{}) \simeq 0 \\ W_{-1}((-1, -1)_R, -) &\simeq \mathbb{Z}/2\mathbb{Z} \end{aligned} \quad (2.1)$$

where $-$ denotes conjugation and $\widehat{}$ denotes the exchange involution, cf. [2, Lemma 2.1 and §3.1].

Proposition 2.2. Let (A, σ) be an F -algebra with involution.

(1) We have

$$\text{Nil}[A, \sigma] := \{P \in X_F \mid W(A \otimes_F F_P, \sigma \otimes \text{id}) \text{ is isomorphic to } 0 \text{ or } \mathbb{Z}/2\mathbb{Z}\}.$$

(2) Let (B, τ) be an F -algebra with involution such that $A \sim B$ and σ and τ are of the same type. Then $\text{Nil}[B, \tau] = \text{Nil}[A, \sigma]$.

(3) Let L be an algebraic extension of F and let $P \in X_F$. For every extension Q of P to L we have $Q \in \text{Nil}[A \otimes_F L, \sigma \otimes \text{id}]$ if and only if $P \in \text{Nil}[A, \sigma]$.

(4) Let $P \in X_F$. Then $P \in \text{Nil}[A, \sigma]$ if and only if any morphism from $W(A \otimes_F F_P, \sigma \otimes \text{id})$ to \mathbb{Z} is identically zero.

Proof. (1) Let $P \in X_F$. The statement follows from considering the list (2.1) with $R = F_P$.

(2) Let $P \in X_F$. By the assumption and Morita theory, $W(A \otimes_F F_P, \sigma \otimes \text{id}) \simeq W(B \otimes_F F_P, \tau \otimes \text{id})$.

Statement (3) follows from the observation that $(A \otimes_F L) \otimes_L L_Q \simeq A \otimes_F F_P$ and (4) follows from [11, Thm. 4.1]. \square

We remark that by Proposition 2.2 our exposition of nil-orderings in this paper is equivalent to those in [2] and [1].

Definition 2.3. Let (D, ϑ) be an F -division algebra with involution and let $P \in X_F$. We say that (D, ϑ) is (F, P) -real (or simply F -real in case F has a unique ordering) if

$$(D, \vartheta) \in \{(F, \text{id}), (F(\sqrt{-d}), -), ((-a, -b)_F, -)\},$$

where $a, b, d \in P$ and $-$ denotes conjugation.

Lemma 2.4. Let (D, ϑ) be an F -division algebra with involution such that $\deg D \leq 2$ and let $P \in X_F$. The following statements are equivalent:

- (1) (D, ϑ) is (F, P) -real;
- (2) $(D \otimes_F F_P, \vartheta \otimes \text{id})$ is F_P -real;
- (3) $P \notin \text{Nil}[D, \vartheta]$.

Proof. (1) \Rightarrow (2) is clear. (2) \Rightarrow (3): $W(D \otimes_F F_P, \vartheta \otimes \text{id})$ is not torsion (cf. [2, §3.1]) and thus $P \notin \text{Nil}[D, \vartheta]$. (3) \Rightarrow (1) follows from an examination of (2.1), Definition 2.3 and Proposition 2.2(1). \square

3 Real Splitting and Maximal Ordered Extensions

In this section, we study the “real splitting” behaviour of an F -algebra with involution (A, σ) . Up to Brauer equivalence, it suffices to consider the underlying F -division algebra D , which is equipped with an involution ϑ of the same kind as σ , as observed in Section 2. Throughout the paper, Z will denote the centre of D . Note that when ϑ is of the first kind, $Z = F$ and the degree of D , $\deg D$, is a 2-power, cf. [14, Cor., p. 154].

Lemma 3.1. Let D be a division algebra with centre Z and let $Z \subseteq K \subseteq M$ be subfields of D , where M is maximal. Then $D \otimes_Z K$ is Brauer equivalent to a quaternion division algebra over K if and only if $[M : K] = 2$.

Proof. Let $k = [K : \mathbb{Z}]$ and $\ell = [M : K]$, then by [14, Chap. 1, §2.9, Prop., p. 139],

$$D \otimes_{\mathbb{Z}} K \simeq C_D(K) \otimes_K M_k(K),$$

where $C_D(K)$ denotes the centralizer of K in D . Thus, $\dim_{\mathbb{Z}} D = \dim_K C_D(K) \cdot k^2$. Also, $D \otimes_{\mathbb{Z}} M \simeq M_{k\ell}(M)$ since M is a splitting field of D , cf. [14, Thm. 2, p. 139]. Thus, $\dim_{\mathbb{Z}} D = (\ell k)^2$. It follows that $\dim_K C_D(K) = \ell^2$ and thus $C_D(K)$ is a quaternion algebra over K if and only if $\ell = 2$. Since $D \otimes_{\mathbb{Z}} K$ is Brauer equivalent to $C_D(K)$, the result follows. \square

Proposition 3.2. *Let (K, P) be an ordered field and $L \supseteq K$ a finite field extension with $[L : K]$ a power of 2 and such that (K, P) has no proper ordered extension in L . Then L and K_P are linearly disjoint over K . In particular, $L \otimes_K K_P$ is a field and $[L : K] \leq 2$.*

Proof. We also use P to denote the set of positive elements of K_P . We know that (see for instance [3, Thm. 1.2.2]) $K_P = \bigcup_{i < \lambda} K_i$ where λ is an ordinal, $K_0 = K$ and, for each $i < \lambda$,

- (1) $K_{i+1} = K_i(\alpha_i)$ where α_i is a square root of an element in $K_i \cap P$ or is a root of a polynomial of odd degree with coefficients in K_i .
- (2) If i is a limit ordinal then $K_i = \bigcup_{j < i} K_j$.

The proof will follow this construction of K_P by transfinite induction.

Fact 3.3. Let $K \subseteq K' \subseteq K_P$ be such that L and K' are linearly disjoint over K . Let $\alpha \in K_P$ be either the square root of an element of $P \cap K'$ or a root of an odd degree polynomial in $K'[X]$. Then every subset of L that is linearly independent over K' is linearly independent over $K'(\alpha)$. In particular L and $K'(\alpha)$ are linearly disjoint over K .

Proof of the Fact: We only prove the first statement. The second one clearly follows from it since L and K' are linearly disjoint over K . We consider two cases, according to α .

Case 1: $\alpha = \sqrt{\beta}$ with $\beta \in K' \cap P$. Let $\ell_1, \dots, \ell_t \in L$ be linearly independent over K' , and assume $a_1 \ell_1 + \dots + a_t \ell_t = 0$ for some $a_1, \dots, a_t \in K'(\alpha)$. Write $a_i = b_i + c_i \sqrt{\beta}$ with $b_i, c_i \in K'$. Then

$$(b_1 \ell_1 + \dots + b_t \ell_t) + (c_1 \ell_1 + \dots + c_t \ell_t) \sqrt{\beta} = 0. \quad (3.1)$$

But $\sqrt{\beta} \notin L$, otherwise we would have $K(\sqrt{\beta}) \subseteq L$, and $K(\sqrt{\beta})$ would be a proper ordered extension of (K, P) in L , contradiction. Since $\sqrt{\beta} \notin L$, its minimal polynomial has degree at least 2 and (3.1) implies $b_1 \ell_1 + \dots + b_t \ell_t = 0$ and $c_1 \ell_1 + \dots + c_t \ell_t = 0$, which yields $b_1 = \dots = b_t = c_1 = \dots = c_t = 0$.

Case 2: α is a root of some polynomial of odd degree in $K'[X]$. Since K' and L are linearly disjoint over K , $L \otimes_K K'$ is a field [12, Prop. 20.2] and $[L \otimes_K K' : K'] = [L : K]$ is a power of 2. Since $[K'(\alpha) : K']$ is odd, the field extensions $L \otimes_K K'$ and $K'(\alpha)$ of K' have relatively prime degrees, so are linearly disjoint over K' [12, Ex. 20.5]. Let $\ell_1, \dots, \ell_t \in L$ be linearly independent over K' . Then ℓ_1, \dots, ℓ_t , considered as elements of $L \otimes_F K' = LK'$, are linearly independent over K' and by the observation above, are linearly independent over $K'(\alpha)$. This concludes the proof of the fact.

Using Fact 3.3, we obtain that if L and K_i are linearly disjoint over K , then L and K_{i+1} are linearly disjoint over K . If $\mu \leq \lambda$ is a limit ordinal, and L and K_i are linearly disjoint over K for every $i < \mu$, then it is immediate that L and $\bigcup_{i < \mu} K_i$ are also linearly disjoint over K .

It follows that $L \otimes_K K_P$ is a field, containing K_P . Since K_P is real closed, $[L \otimes_K K_P : K_P] \leq 2$ and we conclude that $[L : K] \leq 2$. \square

Lemma 3.4. *Let (D, ϑ) be an F -division algebra with involution of the second kind, and let $P \in X_F \setminus \text{Nil}[D, \vartheta]$. Then P does not extend to Z .*

Proof. Assume that P extends to Z . Write $Z = F(\sqrt{\alpha})$ with $\alpha \in P$. Then the centre of $D \otimes_F F_P$ is isomorphic to $Z \otimes_F F_P$ (cf. [13, Lemma 12.4c]) which is isomorphic to $F_P \times F_P$. Therefore $D \otimes_F F_P$ is an F_P -algebra which is not simple. By [2, Lemma 2.1(iv)] $W(D \otimes_F F_P, \vartheta_P) = 0$, implying that $P \in \text{Nil}[D, \vartheta]$, a contradiction. \square

3.1 Involutions of the first kind

Proposition 3.5. *Let D be a central F -division algebra whose degree is a power of two. Let $P \in X_F$ and let (L, Q) be a maximal ordered extension of (F, P) in D . Then one of the following holds*

- (1) L is a maximal subfield of D and $D \otimes_F L \sim L$;
- (2) *There exists $d \in Q$ such that $L(\sqrt{-d})$ is a proper extension of L and a maximal subfield of D . Furthermore, $D \otimes_F L$ is Brauer equivalent to a quaternion division algebra $(-d, -c)_L$ for some $c \in L$.*

Proof. If L is a maximal subfield of D , then $D \otimes_F L$ is Brauer equivalent to L , so we assume from now on that L is not a maximal subfield of D . Let $L \subseteq M$ with M a maximal subfield of D and let L_Q be a real closure of L at Q . Since $[M : L]$ is a power of two, it follows by Proposition 3.2 that $[M : L] = 2$, and thus $M = L(\sqrt{-d})$ for some $d \in Q$. By Lemma 3.1, $D \otimes_F L$ is Brauer equivalent to a quaternion division algebra H over L . Now $D \otimes_F L(\sqrt{-d}) \simeq D \otimes_F L \otimes_L L(\sqrt{-d}) \simeq$

$M_t(H) \otimes_L L(\sqrt{-d}) \simeq M_t(H \otimes_L L(\sqrt{-d}))$ for some $t \in \mathbb{N}$, and since $L(\sqrt{-d})$ is a maximal subfield of D we also have $D \otimes_F L(\sqrt{-d}) \sim L(\sqrt{-d})$, so H splits over $L(\sqrt{-d})$, and the result follows by [10, Chap. III. Thm. 4.1]. \square

Theorem 3.6. *Let (D, ϑ) be an F -division algebra with involution of the first kind. Let $P \in X_F \setminus \text{Nil}[D, \vartheta]$ and let (L, Q) be a maximal ordered extension of (F, P) in D .*

- (1) *If ϑ is orthogonal, then L is a maximal subfield of D and $D \otimes_F L \sim L$;*
- (2) *If ϑ is symplectic, there exists $d \in Q$ such that $L(\sqrt{-d})$ is a proper extension of L and a maximal subfield of D . Furthermore, $D \otimes_F L$ is Brauer equivalent to a quaternion division algebra $(-d, -c)_L$ with $c \in Q$.*

Proof. Since ϑ is of the first kind, $\deg D$ is a power of two.

(1) Let ϑ be orthogonal. Assume that L is not a maximal subfield of D . Then $D \otimes_F L \sim (-d, -c)_L$ for some $c \in L$ and $d \in Q$ by Proposition 3.5. Since ϑ is orthogonal and $P \notin \text{Nil}[D, \vartheta]$, the algebra $D \otimes_F F_P$ is split, which forces $-c \in Q$. Consider the field extension $L(\sqrt{-c})$, which is a proper extension of L since the quaternion algebra $(-d, -c)_L$ is division. The ordering Q extends to an ordering R of $L(\sqrt{-c})$ since $-c \in Q$. We now observe that $[L(\sqrt{-c}) : F] = [L(\sqrt{-d}) : F] = \deg D$, since $L(\sqrt{-d})$ is a maximal subfield of D (cf. [14, Thm. 2, p. 139]), and that

$$D \otimes_F L(\sqrt{-c}) \simeq (D \otimes_F L) \otimes_L L(\sqrt{-c}) \sim (-d, -c)_L \otimes_L L(\sqrt{-c}) \sim L(\sqrt{-c}).$$

Hence, by [13, Cor. 13.3, p. 241], $L(\sqrt{-c})$ is F -isomorphic to a subfield K of D . We denote this isomorphism by f . Observe that $f(R)$ is an ordering of K . By the Skolem-Noether theorem (cf. [13, p. 230]), there exists $a \in D^\times$ such that $f|_L = \text{Int}(a)|_L$. It follows that $\text{Int}(a^{-1})(K)$ is a subfield of D and a proper extension of L . Furthermore, the ordering $\text{Int}(a^{-1})(f(R))$ of $\text{Int}(a^{-1})(K)$ extends $\text{Int}(a^{-1})(f(Q)) = Q$, contradicting the choice of (L, Q) as a maximal ordered extension of (F, P) in D . We conclude that L is a maximal subfield of D and that $D \otimes_F L \sim L$.

(2) Let ϑ be symplectic. Assume that L is a maximal subfield of D . Then $D \otimes_F L \sim L$. Thus $\vartheta \otimes \text{id}$ is a symplectic involution on the split algebra $D \otimes_F F_P$, which implies that $W(D \otimes_F F_P, \vartheta \otimes \text{id}) \simeq W_{-1}(F_P, \text{id})$. It follows from Proposition 2.2(1) that $P \in \text{Nil}[D, \vartheta]$, a contradiction. Thus, by Proposition 3.5, $D \otimes_F L \sim (-d, -c)_L$ for some $c \in L$ and $d \in Q$. As above, since $P \notin \text{Nil}[D, \vartheta]$, $D \otimes_F F_P$ is not split, forcing $c \in Q$. \square

3.2 Involutions of the second kind: the 2-power degree case

Lemma 3.7. *Let D be an F -division algebra with centre Z such that $[Z : F] = 2$. Assume that the degree of D is a power of 2. Let $P \in X_F$ be such that P does not*

extend to Z and let (L, Q) be a maximal ordered extension of (F, P) in D . Then $D \otimes_F L$ is Brauer equivalent to $L(\sqrt{-d})$ for some $d \in P$ with $\sqrt{-d} \notin L$.

Proof. By hypothesis, $Z = F(\sqrt{-d})$ for some $d \in P$. It follows that $L(\sqrt{-d})$ is a proper field extension of L . We claim that $L(\sqrt{-d})$ is a maximal subfield of D . Indeed, let N be a maximal subfield of D such that $L(\sqrt{-d}) \subseteq N$. Since $\dim_Z D$ is a power of 2 and (L, Q) has no ordered extension in N , we can apply Proposition 3.2, and we obtain $[N : L] = 2$, so $N = L(\sqrt{-d})$ is a maximal subfield of D . Therefore, we have $D \otimes_F L \simeq (D \otimes_Z Z) \otimes_F L \simeq D \otimes_Z (F(\sqrt{-d}) \otimes_F L) \simeq D \otimes_Z L(\sqrt{-d}) \sim L(\sqrt{-d})$. \square

Theorem 3.8. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Assume that the degree of D is a power of 2. Let $P \in X_F \setminus \text{Nil}[D, \vartheta]$ and let (L, Q) be a maximal ordered extension of (F, P) in D . Then $D \otimes_F L$ is Brauer equivalent to $L(\sqrt{-d})$ for some $d \in P$ with $\sqrt{-d} \notin L$.*

Proof. By Lemma 3.4, P does not extend to the centre of D and we conclude with Lemma 3.7. \square

Proposition 3.9. *Let D be an F -division algebra with centre Z such that $[Z : F] = 2$. Assume that the degree of D is a power of 2. Let $P \in X_F$ be such that P extends to an ordering P' of Z and let (L, Q) be a maximal ordered extension of (Z, P') in D . Then $D \otimes_F L$ is isomorphic to $M_r(L \times L)$ or to $M_s((a, b)_L \times (a, b)_L)$ for some $r, s \in \mathbb{N}$ and $a, b \in L^\times$.*

Proof. Let $\alpha \in P$ be such that $Z = F(\sqrt{\alpha})$. Let N be a maximal field extension of L in D . By Proposition 3.2, $[N : L] = 1$ or 2 . Hence $D \otimes_Z L$ is isomorphic to either $M_r(L)$ or $M_s((a, b)_L)$ for some $r, s \in \mathbb{N}$ and $a, b \in L^\times$ by Lemma 3.1. Then

$$\begin{aligned} D \otimes_F L &\simeq (D \otimes_Z Z) \otimes_F L \\ &\simeq D \otimes_Z (Z \otimes_F L) \\ &\simeq D \otimes_Z (F[x]/(x^2 - \alpha) \otimes_F L) \\ &\simeq D \otimes_Z L[x]/(x^2 - \alpha) \\ &\simeq D \otimes_Z (L \times L) \\ &\simeq (D \otimes_Z L) \times (D \otimes_Z L) \end{aligned}$$

and the result follows. \square

3.3 Involutions of the second kind: the odd degree case

Lemma 3.10. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Let $P \in X_F$ be such that P does not extend to Z and write $Z = F(\sqrt{-d})$*

with $\vartheta(\sqrt{-d}) = -\sqrt{-d}$ and $d \in P$. Let L be a field extension of F in D such that $\sqrt{-d} \notin L$. Then there is an F -linear involution $\sigma : D \rightarrow D$ such that σ is the identity on L and $\sigma(\sqrt{-d}) = -\sqrt{-d}$.

Proof. We consider

$$\xi : L(\sqrt{-d}) \longrightarrow D, \quad \ell_1 + \ell_2 \sqrt{-d} \longmapsto \vartheta(\ell_1) + \vartheta(\ell_2) \sqrt{-d}.$$

A direct computation shows that ξ is a morphism of rings and is Z -linear, so a morphism of Z -algebras from the simple Z -algebra $L(\sqrt{-d})$ to the central simple Z -algebra D . By the Skolem-Noether theorem (see [13, p. 230]) there is $a \in D^\times$ such that $\xi(x) = a^{-1}xa$ for every $x \in L(\sqrt{-d})$.

We define $\sigma := \text{Int}(a^{-1}) \circ \vartheta$. This is an involution on D , and, for $\ell \in L$, $\sigma(\ell) = a\vartheta(\ell)a^{-1} = a\xi(\ell)a^{-1} = aa^{-1}\ell aa^{-1} = \ell$, so σ is the identity on L . Finally $\sigma(\sqrt{-d}) = a\vartheta(\sqrt{-d})a^{-1} = a(-\sqrt{-d})a^{-1} = -\sqrt{-d}$ (since $\sqrt{-d}$ is in the centre of D). \square

Theorem 3.11. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Assume that the degree of D is odd. Let $P \in X_F \setminus \text{Nil}[D, \vartheta]$, and write $Z = F(\sqrt{-d})$ with $\vartheta(\sqrt{-d}) = -\sqrt{-d}$ and $d \in P$ (cf. Lemma 3.4). Let (L, Q) be a maximal ordered extension of (F, P) in D . Then $L(\sqrt{-d})$ is a maximal subfield of D and $D \otimes_F L \sim L(\sqrt{-d})$.*

Proof. Since (L, Q) is an ordered extension of (F, P) , we have $\sqrt{-d} \notin L$, and by Lemma 3.10 there is an F -involution σ on D such that σ is the identity on L and $\sigma(\sqrt{-d}) = -\sqrt{-d}$. The following is inspired by an argument in the proof of [9, Prop. 11.22]. Let $M = L(\sqrt{-d})$, and assume that M is not a maximal subfield of D .

Claim: Then there is $x \in C_D(M) \setminus M$ such that $\sigma(x) = x$.

Proof of the claim: By construction of σ we have $\sigma(M) = M$ and it follows easily that $\sigma(C_D(M)) \subseteq C_D(M)$. Since M is not maximal, there is $u \in C_D(M) \setminus M$, cf. [13, p. 236, Cor. b]. Let $u_1 = (u + \sigma(u))/2$ and $u_2 = (u - \sigma(u))/2$. Since $\sigma(C_D(M)) \subseteq C_D(M)$ we have $u_1, u_2 \in C_D(M)$. Obviously $u = u_1 + u_2$, with $\sigma(u_1) = u_1$ and $\sigma(u_2) = -u_2$. If $u_1 \notin M$, then we can take $x = u_1$. If $u_1 \in M$, then $u_2 \notin M$, and we can take $x = u_2 \sqrt{-d}$, which establishes the claim.

Consider the field $L(x)$. By choice of x we have $L \subsetneq L(x) \subseteq \text{Sym}(D, \sigma)$. In particular $\sqrt{-d} \notin L(x)$, and the diagram

$$\begin{array}{ccc} & L(x, \sqrt{-d}) & \\ \begin{array}{c} \nearrow 2 \\ \nwarrow \end{array} & & \begin{array}{c} \nwarrow k \text{ odd} \\ \nearrow \end{array} \\ L(x) & & Z = F(\sqrt{-d}) \\ \searrow & & \nearrow 2 \\ & F & \end{array}$$

gives us that $[L(x) : F] = k$ is odd. It follows that $[L(x) : L]$ is odd, and in particular the ordering Q of L extends to $L(x)$, a contradiction to the choice of (L, Q) . Thus $L(\sqrt{-d})$ is a maximal subfield of D and $D \otimes_F L \simeq D \otimes_Z (Z \otimes_F L) \simeq D \otimes_Z L(\sqrt{-d}) \sim L(\sqrt{-d})$. \square

Proposition 3.12. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Assume that the degree of D is odd. Let $P \in X_F$ be such that P extends to Z and let L be a maximal subfield of D . Then P extends to L and $D \otimes_F L \simeq M_r(L \times L)$ for some $r \in \mathbb{N}$.*

Proof. Since P extends to Z , we have that $Z = F(\sqrt{\alpha})$ for some $\alpha \in P$. Since $[L : Z]$ is odd, P extends to L . Also, $D \otimes_Z L \simeq M_r(L)$ for some $r \in \mathbb{N}$ since L is maximal in D . Then

$$\begin{aligned} D \otimes_F L &\simeq (D \otimes_Z Z) \otimes_F L \\ &\simeq D \otimes_Z (F(\sqrt{\alpha}) \otimes_F L) \\ &\simeq D \otimes_Z (F[x]/(x^2 - \alpha) \otimes_F L) \\ &\simeq D \otimes_Z L[x]/(x^2 - \alpha) \\ &\simeq D \otimes_Z (L \times L) \\ &\simeq M_r(L) \times M_r(L). \end{aligned} \quad \square$$

3.4 Involutions of the second kind: the arbitrary degree case

Theorem 3.13. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Let $P \in X_F$ be such that P does not extend to Z and write $Z = F(\sqrt{-d})$ for some $d \in P$. There exists a maximal ordered extension (L, Q) of (F, P) in D such that $L(\sqrt{-d})$ is a maximal subfield of D and $D \otimes_F L \sim L(\sqrt{-d})$.*

Proof. We may write $D \simeq D_1 \otimes_Z D_2$ where D_1 is an F -division algebra of odd degree with centre Z and D_2 is an F -division algebra of 2-power degree with centre Z , cf. [13, Primary Decomposition Theorem, p. 261]. (Note though that the involution ϑ is not necessarily decomposable.)

Let (L_2, P') be a maximal ordered extension of (F, P) inside D_2 . By Theorem 3.8, $L_2(\sqrt{-d})$ is a maximal subfield of D_2 and

$$D_2 \otimes_F L_2 \simeq D_2 \otimes_Z L_2(\sqrt{-d}) \simeq M_k(L_2(\sqrt{-d}))$$

for some nonzero integer k . Hence

$$\begin{aligned} D \otimes_F L_2 &\simeq D_1 \otimes_Z M_k(L_2(\sqrt{-d})) \\ &\simeq D_1 \otimes_Z M_k(L_2 \otimes_F F(\sqrt{-d})) \end{aligned}$$

$$\begin{aligned}
&\simeq D_1 \otimes_Z M_k(\underbrace{F(\sqrt{-d})}_Z) \otimes_F L_2 \\
&\simeq M_k(D_1) \otimes_F L_2 \\
&\simeq M_k(D_1 \otimes_F L_2).
\end{aligned}$$

Since $[L_2(\sqrt{-d}) : Z]$ is coprime to $\deg D_1$ and $D_1 \otimes_F L_2 \simeq D_1 \otimes_Z L_2(\sqrt{-d})$, we have that $D_1 \otimes_F L_2$ is still a division algebra (cf. [13, Prop. 13.4(vi), p. 243]), which is of (odd) degree $\deg D_1$ over its centre $L_2(\sqrt{-d})$ (cf. [13, Lemma 12.4c, p. 225]). Since it is Brauer equivalent to $D \otimes_F L_2$ it possesses an involution τ of the second kind with fixed field L_2 (cf. [9, Thm. 3.1]).

Let (L, Q) be a maximal ordered extension of (L_2, P') inside $D_1 \otimes_F L_2$. By Theorem 3.11, $L(\sqrt{-d})$ is a maximal subfield of $D_1 \otimes_F L_2$ and

$$D_1 \otimes_F L \simeq (D_1 \otimes_F L_2) \otimes_{L_2} L \sim L(\sqrt{-d}).$$

Hence

$$\begin{aligned}
D \otimes_F L &\simeq (D \otimes_F L_2) \otimes_{L_2} L \\
&\simeq M_k(D_1 \otimes_F L_2) \otimes_{L_2} L \\
&\simeq M_k(D_1 \otimes_F L) \\
&\sim L(\sqrt{-d}).
\end{aligned}$$

Note that L is a subfield of D since $L \subseteq D_1 \otimes_F L_2 \simeq D_1 \otimes_Z L_2(\sqrt{-d}) \subseteq D_1 \otimes_Z D_2 \simeq D$. Thus $L(\sqrt{-d})$ is a subfield of D since $\sqrt{-d} \in D$. Finally,

$$[L(\sqrt{-d}) : Z] = [L(\sqrt{-d}) : L_2(\sqrt{-d})][L_2(\sqrt{-d}) : Z] = \deg D_1 \cdot \deg D_2 = \deg D$$

since $L(\sqrt{-d})$ is a maximal subfield of $D_1 \otimes_F L_2 \simeq D_1 \otimes_Z L_2(\sqrt{-d})$ and $L_2(\sqrt{-d})$ is a maximal subfield of D_2 . Hence $L(\sqrt{-d})$ is a maximal subfield of D by [14, Cor., p. 139]. It follows that (L, Q) is a maximal ordered extension of (F, P) inside D , for if (M, R) were any proper ordered extension of (L, Q) , M would be maximal and would contain Z , which is a contradiction since $d \in P$ and $\sqrt{-d} \in Z$. \square

Corollary 3.14. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Let $P \in X_F \setminus \text{Nil}[D, \vartheta]$ and write $Z = F(\sqrt{-d})$ for some $d \in P$ (cf. Lemma 3.4). There exists a maximal ordered extension (L, Q) of (F, P) in D such that $L(\sqrt{-d})$ is a maximal subfield of D and $D \otimes_F L \sim L(\sqrt{-d})$.*

Proof. By Lemma 3.4, P does not extend to the centre of D and we conclude with Theorem 3.13. \square

Proposition 3.15. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Let $P \in X_F$ be such that P extends to an ordering P' on Z . There exists a maximal ordered extension (L, Q) of (Z, P') in D such that $D \otimes_F L$ is isomorphic to $M_r(L \times L)$ or to $M_s((a, b)_L \times (a, b)_L)$ for some $r, s \in \mathbb{N}$ and $a, b \in L^\times$.*

Proof. We may write $D \simeq D_1 \otimes_Z D_2$ where D_1 is an F -division algebra of odd degree with centre Z and D_2 is an F -division algebra of 2-power degree with centre Z . Let L_1 be a maximal subfield of D_1 . Then $[L_1 : Z]$ is odd. Let (L_2, Q_2) be a maximal ordered field extension of (Z, P') in D_2 . By Proposition 3.9 and its proof, L_2 is either a maximal subfield of D_2 or a subfield of index 2 in a maximal subfield of D_2 . Since $[L_1 : Z]$ and $[L_2 : Z]$ are relatively prime, $L_1 \otimes_Z L_2$ is a field and thus a subfield of $D_1 \otimes_Z D_2$. Since it is an extension of odd degree of $1 \otimes_Z L_2$, it carries an ordering Q_3 that extends Q_2 . Let (L, Q) be a maximal ordered field extension of $(L_1 \otimes_Z L_2, Q_3)$ in D . If L_2 is maximal, then $L = L_1 \otimes_Z L_2$ is a maximal subfield of D since $[L : Z] = \deg D$. Otherwise, L has index at most 2 in a maximal subfield of D . Thus $D \otimes_Z L \simeq M_r(L)$ (when L is maximal) or $D \otimes_Z L \simeq M_s((a, b)_L)$ for certain $a, b \in L^\times$ (otherwise) by Lemma 3.1. Then

$$\begin{aligned} D \otimes_F L &\simeq D \otimes_Z (Z \otimes_F L) \\ &\simeq D \otimes_Z (L \times L) \\ &\simeq (D \otimes_Z L) \times (D \otimes_Z L) \end{aligned}$$

and the result follows. \square

Theorem 3.16. *Let (D, ϑ) be an F -division algebra with involution of the second kind. Let $\alpha \in F$ be such that $Z = F(\sqrt{\alpha})$. Then*

$$\text{Nil}[D, \vartheta] = H(\alpha) = \{P \in X_F \mid P \text{ extends to } Z\},$$

where $H(\alpha)$ denotes the usual Harrison set.

Proof. We will show that $X_F \setminus \text{Nil}[D, \vartheta] = H(-\alpha)$. By Lemma 3.4, $P \notin \text{Nil}[D, \vartheta]$ implies $P \in H(-\alpha)$. Now let $P \in \text{Nil}[D, \vartheta]$. Assume $P \in H(-\alpha)$. Then P does not extend to Z . Let (L, Q) be as in Theorem 3.13 and note that $\alpha \notin Q$. Then $D \otimes_F L \simeq M_r(L(\sqrt{\alpha}))$ for some $r \in \mathbb{N}$. Let L_Q be a real closure of L at Q . Then $\sqrt{\alpha} \notin L_Q$ and

$$\begin{aligned} D \otimes_F L_Q &\simeq (D \otimes_F L) \otimes_L L_Q \\ &\simeq M_r(L(\sqrt{\alpha}) \otimes_L L_Q) \\ &\simeq M_r(L_Q(\sqrt{\alpha})) \\ &\simeq M_r(L_Q(\sqrt{-1})). \end{aligned}$$

Since $F_P \simeq L_Q$, it follows that $D \otimes_F F_P \simeq M_r(F_P(\sqrt{-1}))$ and thus from (2.1) and Proposition 2.2(1) that $P \notin \text{Nil}[D, \vartheta]$. \square

Note that therefore $\text{Nil}[D, \vartheta]$ is clopen in X_F . This was already proved in a different way in [2, Cor. 6.5].

We finish this section with the following natural question: Let (D, ϑ) be an F -division algebra with involution of the second kind, let $P \in X_F$ and let (L, Q) be any maximal ordered extension of (F, P) in D . Do the conclusions of Theorem 3.13 and Proposition 3.15 hold for $D \otimes_F L$? We are currently unable to provide an answer.

4 H -Signatures Revisited

Using results obtained in [2] and [1] we give a self-contained presentation of H -signatures of hermitian forms in the following paragraphs. Let (A, σ) be an F -algebra with involution and let $P \in \widetilde{X}_F$. Using Proposition 2.2(1), Lemma 2.4 and (2.1) we obtain the sequence of group morphisms

$$W(A, \sigma) \xrightarrow{r_P} W(A \otimes_F F_P, \sigma \otimes \text{id}) \xrightarrow[\simeq]{\mu_P} W(D_P, \vartheta_P) \xrightarrow{\rho_P} W(F_P) \xrightarrow{\text{sign}} \mathbb{Z}, \quad (4.1)$$

where r_P is the canonical restriction map, (D_P, ϑ_P) is an F_P -real division algebra with involution, μ_P is an isomorphism induced by Morita theory, ρ_P is defined by $\rho_P(\eta)(x) := \eta(x, x)$ for all $\eta \in W(D_P, \vartheta_P)$ (cf. [7]) and sign is the usual signature of quadratic forms at the unique ordering of F_P .

In [2, §3.2] we showed that the map $|\text{sign} \circ \rho_P \circ \mu_P \circ r_P|$ does not depend on the choice of F_P and μ_P . In [1, Prop. 3.2] we showed that there exists a hermitian form $H_0 \in W(A, \sigma)$ such that for all $P \in \widetilde{X}_F$,

$$\text{sign}(\rho_P \circ \mu_P)(H_0 \otimes F_P) \neq 0. \quad (4.2)$$

We call H_0 a *reference form* for (A, σ) .

Definition 4.1. Let $H_0 \in W(A, \sigma)$ be a reference form for (A, σ) , let $h \in W(A, \sigma)$, let $P \in \widetilde{X}_F$, let $\ell = \dim_{F_P} D_P$ and let $\delta_P^{H_0} = \text{sgn}(\text{sign}(\rho_P \circ \mu_P)(H_0 \otimes F_P)) \in \{-1, 1\}$. We define the H -signature of h at $P \in X_F$, $\text{sign}_P^{H_0} h$, by

$$\text{sign}_P^{H_0} h := \frac{1}{\ell} \delta_P^{H_0} \text{sign}(\rho_P \circ \mu_P)(h \otimes F_P)$$

whenever $P \in \widetilde{X}_F$ and $\text{sign}_P^{H_0} h := 0$ if $P \in \text{Nil}[A, \sigma]$, cf. Proposition 2.2(4).

Remark 4.2.

- (1) We showed in [2, §3.3] that $\text{sign}_P^{H_0}$ only depends on P and H_0 and in [1, §7] that H -signatures correspond canonically to a natural class of morphisms from $W(A, \sigma)$ to \mathbb{Z} .

- (2) If L is a finite extension of F , it follows from (4.2) that $H_0 \otimes_F L$ is a reference form for $(A \otimes_F L, \sigma \otimes \text{id})$. Moreover, if R is an ordering on L that extends $P \in X_F$, then

$$\text{sign}_R^{H_0 \otimes L}(h \otimes L) = \text{sign}_P^{H_0} h$$

for all $h \in W(A, \sigma)$.

- (3) Let $P \in \widetilde{X}_F$. If H_1 is another reference form for (A, σ) , then easy computations show that

$$\text{sign}_P^{H_0} h = \delta_P^{H_0} \delta_P^{H_1} \text{sign}_P^{H_1} h$$

for all $h \in W(A, \sigma)$ and

$$\delta_P^{H_0} \delta_P^{H_1} = \text{sgn}(\text{sign}_P^{H_1} H_0) = \text{sgn}(\text{sign}_P^{H_0} H_1).$$

Lemma 4.3. *Let $P \in X_F$, let (D, ϑ) be an F -division algebra with involution which is (F, P) -real and let $\ell = \dim_F D$. Consider the reference form $H = \langle 1 \rangle_{\vartheta}$ and the group morphism $\rho : W(D, \vartheta) \longrightarrow W(F)$, where $\rho(h)(x) := h(x, x)$. Then*

$$\text{sign}_P^H h = \frac{1}{\ell} \text{sign}_P \rho(h).$$

Proof. Observe that H is indeed a reference form. Let F_P denote a real closure of F at P and note that $P \notin \text{Nil}[D, \vartheta]$ by Lemma 2.4. Consider the diagram

$$\begin{array}{ccccc} W(D, \vartheta) & \xrightarrow{r_P} & W(D \otimes_F F_P, \vartheta \otimes \text{id}) & \xrightarrow{\text{id}} & W(D \otimes_F F_P, \vartheta \otimes \text{id}) \\ \rho \downarrow & & & & \downarrow \rho_P \\ W(F) & \xrightarrow{r'_P} & W(F_P) & \xrightarrow{\text{sign}} & \mathbb{Z} \end{array} \quad (4.3)$$

where we used the notation from the sequence (4.1). The square on the left commutes: Let $h \in W(D, \vartheta)$. Then $\rho(h) \otimes F_P = \rho_P(h \otimes F_P)$ since

$$(\rho(h) \otimes F_P)(x \otimes 1) = \rho(h)(x) \otimes 1 = h(x, x) \otimes 1 = (h \otimes F_P)(x \otimes 1, x \otimes 1) = \rho_P(h \otimes F_P)(x \otimes 1).$$

It follows that

$$\begin{aligned} \text{sign}_P^H h &= \frac{1}{\ell} \varepsilon_P \text{sign}(\rho_P \circ \text{id})(h \otimes F_P) \\ &= \frac{1}{\ell} \text{sign} \rho(h) \otimes F_P \\ &= \frac{1}{\ell} \text{sign}_P \rho(h), \end{aligned}$$

where $\varepsilon_P = \text{sgn}(\text{sign}(\rho_P \circ \text{id})(H \otimes F_P)) = \text{sgn}(\text{sign}(\ell \times \langle 1 \rangle)) = 1$. \square

Lemma 4.4. *Let $P \in \widetilde{X}_F$ and let (D, ϑ) be an (F, P) -real division algebra with involution. Let H be a reference form for (D, ϑ) . The signature map*

$$\text{sign}_P^H : W(D, \vartheta) \longrightarrow \mathbb{Z}$$

is surjective.

Proof. Let $h = \langle 1 \rangle_\vartheta$. Then $\text{sign}_P^H h = \pm 1$ by Remark 4.2(3) and Lemma 4.3. \square

5 Stability Index of Algebras with Involution

In this section, we fix an F -algebra with involution (A, σ) and a reference form H for (A, σ) . Let $C(X_F, \mathbb{Z})$ denote the ring of continuous functions from X_F (equipped with the Harrison topology) to \mathbb{Z} (equipped with the discrete topology). For every $h \in W(A, \sigma)$ we denote by $\text{sign}^H h$ the total signature map from X_F to \mathbb{Z} and remark that it is a continuous map, cf. [2, Thm. 7.2]. If $h \in W(A, \sigma)$, we have $\text{sign}^H h = 0$ on $\text{Nil}[A, \sigma]$. Therefore it is convenient to introduce the notation

$$\widetilde{C}(X_F, \mathbb{Z}) := \{f \in C(X_F, \mathbb{Z}) \mid f = 0 \text{ on } \text{Nil}[A, \sigma]\}.$$

Note that $\widetilde{C}(X_F, \mathbb{Z})$ depends on the Brauer class of A and the type of σ , but indicating this would make the notation cumbersome. Since (A, σ) is fixed, no confusion should arise. For $P \in X_F$ and a field extension L of F , we define

$$X_L/P := \{Q \in X_L \mid P \subseteq Q\}.$$

Lemma 5.1. *Let $P \in \widetilde{X}_F$. Then there exists a hermitian form $h_P \in W(A, \sigma)$ and a positive integer ℓ_P such that $\text{sign}_P^H h_P = 2^{\ell_P}$.*

Proof. By Theorem 3.6 and Corollary 3.14 there exists an ordered field extension (L, Q) of (F, P) with $L \subseteq A$ and such that $(A \otimes_F L, \sigma \otimes \text{id})$ is Morita equivalent to an (L, Q) -real division algebra with involution. Note that $[L : F]$ is finite. Let $Q_0 \in X_L/P$. By Lemma 4.4 and [1, Thm. 4.2] there exists a hermitian form h over $(A \otimes_F L, \sigma \otimes \text{id})$ such that $\text{sign}_{Q_0}^{H \otimes L} h = 1$. Since X_L/P is finite, there exist $a_1, \dots, a_r \in L$ such that $\{Q_0\} = \{Q \in X_L/P \mid a_1, \dots, a_r \in Q\}$. Using the notation from [2, §5.3], let $h_P := \text{Tr}_{A \otimes_F L}^*(\langle\langle a_1, \dots, a_r \rangle\rangle \cdot h)$. It follows from the Knebusch trace formula [2, Thm. 8.1] that

$$\begin{aligned} \text{sign}_P^H h_P &= \sum_{Q \in X_L/P} \text{sign}_Q^{H \otimes L}(\langle\langle a_1, \dots, a_r \rangle\rangle \cdot h) \\ &= 2^r \text{sign}_{Q_0}^{H \otimes L} h \\ &= 2^r. \end{aligned} \quad \square$$

Lemma 5.2. *There exists a hermitian form $h_0 \in W(A, \sigma)$ and a positive integer k_0 such that $\text{sign}_P^H h_0 = 2^{k_0}$ for every $P \in \widetilde{X}_F$.*

Proof. By Lemma 5.1, for every $P \in \widetilde{X}_F$ there exists $\ell_P \in \mathbb{N}$, U_P clopen in \widetilde{X}_F containing P , and $h_P \in W(A, \sigma)$ such that $\text{sign}^H h_P = 2^{\ell_P}$ on U_P (simply take $U_P = (\text{sign}^H h_P)^{-1}(2^{\ell_P})$).

Therefore $\widetilde{X}_F = \bigcup_{P \in \widetilde{X}_F} U_P = \bigcup_{i=1}^n U_{P_i}$ since \widetilde{X}_F is compact. By removing the intersections of the sets U_{P_i} we obtain $\widetilde{X}_F = \bigcup_{i=1}^r C_i$ where each C_i is clopen and $\text{sign}^H \eta_i = 2^{\ell_i}$ on C_i for hermitian forms $\eta_i \in W(A, \sigma)$, $i = 1, \dots, r$.

Let $q_i \in W(F)$ be such that $\text{sign} q_i$ is equal to 2^{s_i} on C_i (for some integer s_i) and 0 elsewhere, cf. [10, VIII, Lemma 6.10]. Then $\text{sign}^H(q_i \cdot \eta_i)$ is equal to $2^{s_i + \ell_i}$ on C_i and 0 elsewhere. Taking $k_0 = \max\{s_1 + \ell_1, \dots, s_r + \ell_r\}$ and multiplying $q_i \cdot \eta_i$ by a suitable power of 2, we obtain a form $h_i \in W(A, \sigma)$ such that $\text{sign}^H h_i = 2^{k_0}$ on C_i and 0 elsewhere. It follows that, for $h_0 := h_1 + \dots + h_r$, $\text{sign}^H h_0 = 2^{k_0}$ on \widetilde{X}_F . \square

Proposition 5.3. *Let $k_0 \in \mathbb{N}$ and $h_0 \in W(A, \sigma)$ be such that $\text{sign}^H h_0 = 2^{k_0}$ on \widetilde{X}_F .*

- (1) *Let $q \in W(F)$. Then there exists $h \in W(A, \sigma)$ such that $\text{sign}(2^{k_0} q) = \text{sign}^H(h)$ on \widetilde{X}_F .*
- (2) *Let $f \in \widetilde{C}(X_F, \mathbb{Z})$. Then there exists $n \in \mathbb{N}$ such that $2^n f \in \text{Im}(\text{sign}^H)$.*

Proof. (1) We take $h = q \cdot h_0$.

(2) We know that there exists $m \in \mathbb{N}$ such that $2^m f = \text{sign}(q)$ for some $q \in W(F)$, cf. [10, VIII, Lemma 6.9]. Then $2^{m+k_0} f = \text{sign}^H(q \cdot h_0)$. \square

Definition 5.4.

- (1) We define $\text{st}^H(A, \sigma)$ to be the smallest nonnegative integer k such that

$$2^k \cdot \widetilde{C}(X_F, \mathbb{Z}) \subseteq \text{Im}(\text{sign}^H)$$

if such an integer exists, and infinity otherwise.

- (2) We define $S^H(A, \sigma)$ to be the cokernel of the total signature map

$$\text{sign}^H : W(A, \sigma) \rightarrow \widetilde{C}(X_F, \mathbb{Z}).$$

The following corollary follows immediately from Proposition 5.3.

Corollary 5.5. *The group $S^H(A, \sigma)$ is 2-primary torsion, and its exponent is $2^{\text{st}(A, \sigma)}$ (with the convention that $2^\infty = \infty$).*

Proposition 5.6. *Let H' be another reference form for (A, σ) . Then $S^H(A, \sigma) \simeq S^{H'}(A, \sigma)$. In particular $\text{st}^H(A, \sigma) = \text{st}^{H'}(A, \sigma)$.*

Proof. By [1, Prop. 3.3(iii)], there exists $f \in C(X_F, \{-1, 1\})$ such that $\text{sign}^H = f \cdot \text{sign}^{H'}$. Define

$$\begin{aligned} \xi : \widetilde{C}(X_F, \mathbb{Z}) &\longrightarrow \widetilde{C}(X_F, \mathbb{Z}) / \text{Im } \text{sign}^H \\ g &\longmapsto f \cdot g + \text{Im } \text{sign}^H \end{aligned}$$

The map ξ is a surjective morphism of groups since f is invertible in $C(X_F, \mathbb{Z})$. Moreover, $g \in \ker \xi$ if and only if $f \cdot g \in \text{Im } \text{sign}^H = f \cdot \text{Im } \text{sign}^{H'}$, so $\ker \xi = \text{Im } \text{sign}^{H'}$ and ξ induces an isomorphism from $S^{H'}(A, \sigma)$ to $S^H(A, \sigma)$. \square

Definition 5.7. We call $S(A, \sigma)$ the *stability group* of (A, σ) . It is well-defined up to isomorphism by Proposition 5.6. We call $\text{st}(A, \sigma)$ the *stability index* of (A, σ) .

Proposition 5.8. *Let $h_0 \in W(A, \sigma)$ and $k_0 \in \mathbb{N}$ be as in Lemma 5.2. Then*

$$\text{st}(A, \sigma) \leq \text{st}(F) + k_0.$$

Proof. Assume that $\text{st}(F)$ is finite. Let $f \in \widetilde{C}(X_F, \mathbb{Z})$. Then there exists $q \in W(F)$ such that $2^{\text{st}(F)} f = \text{sign } q$, and thus $\text{sign}^H(q \cdot h_0) = 2^{\text{st}(F)+k_0} f$. \square

Proposition 5.9. *Let (A, σ) and (B, τ) be two Morita equivalent F -algebras with involution. Then $S(A, \sigma) \simeq S(B, \tau)$ and $\text{st}(A, \sigma) = \text{st}(B, \tau)$.*

Proof. It suffices to prove the first part of the statement, but this follows immediately from [1, Thm. 4.2]. \square

The following theorem extends a well-known result in quadratic form theory (cf. [4, (1.6)]) to algebras with involution:

Theorem 5.10. *Let (A, σ) be an F -algebra with involution and let $W_t(A, \sigma)$ denote the torsion subgroup of $W(A, \sigma)$. The sequence*

$$0 \longrightarrow W_t(A, \sigma) \longrightarrow W(A, \sigma) \xrightarrow{\text{sign}^H} \widetilde{C}(X_F, \mathbb{Z}) \longrightarrow S(A, \sigma) \longrightarrow 0$$

is exact. The groups $W_t(A, \sigma)$ and $S(A, \sigma)$ are 2-primary torsion groups.

Proof. This follows from [11, Thm. 4.1], [15, Thm. 5.1] and the definition of $S(A, \sigma)$. \square

Examples 5.11. In each of the examples below, A is a quaternion division algebra.

- (1) Let $F = \mathbb{R}$, $A = (-1, -1)_F$ and σ quaternion conjugation. Then $\text{Im } \text{sign}^H \simeq \mathbb{Z} \simeq \widetilde{C}(X_F, \mathbb{Z})$. Hence $S(A, \sigma) \simeq \{0\}$ and $\text{st}(A, \sigma) = 0$. Note that $\text{st}(\mathbb{R}) = 0$.

- (2) Let $F = \mathbb{R}$, $A = (-1, -1)_F$ and σ orthogonal. Then $\text{Im sign}^H \simeq \{0\} \simeq \widetilde{C}(X_F, \mathbb{Z})$. Hence $S(A, \sigma) \simeq \{0\}$ and $\text{st}(A, \sigma) = 0$.
- (3) Let $F = \mathbb{Q}(\sqrt{2})$, $A = (-1, -\sqrt{2})_F$ and σ quaternion conjugation. Then $\text{Im sign}^H \simeq \mathbb{Z} \times \{0\} \simeq \widetilde{C}(X_F, \mathbb{Z})$. Hence $S(A, \sigma) \simeq \{0\}$ and $\text{st}(A, \sigma) = 0$. Note that $\text{st}(\mathbb{Q}(\sqrt{2})) = 1$.
- (4) Let $F = \mathbb{R}((x))$, $A = (x, -1)_F$ and σ orthogonal. Then $\text{Im sign}^H \simeq 2\mathbb{Z} \times \{0\}$ and $\widetilde{C}(X_F, \mathbb{Z}) \simeq \mathbb{Z} \times \{0\}$. Hence $S(A, \sigma) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\text{st}(A, \sigma) = 1$. Note that $\text{st}(\mathbb{R}((x))) = 1$.

We now consider the total signature $\text{sign}^H h$ of a hermitian form $h \in W(A, \sigma)$. Since this is a continuous function, there exists an integer k such that $2^k \text{sign}^H(h)$ is the total signature of some quadratic form over F . In the next two results we will show that k can be chosen independently of h .

Lemma 5.12. *There exist disjoint clopen subsets U_1, \dots, U_t of \widetilde{X}_F and positive integers n_1, \dots, n_t such that $\widetilde{X}_F = U_1 \cup \dots \cup U_t$ and for every $h \in W(A, \sigma)$ and $i \in \{1, \dots, t\}$, there exists $q_i \in W(F)$ such that $\text{sign}(q_i) = 2^{n_i} \text{sign}^H(h)$ on U_i .*

Proof. By Theorem 3.6 and Corollary 3.14, for every $P \in \widetilde{X}_F$ there exists a finite ordered field extension (L_P, R_P) of (F, P) such that $(A \otimes_F L_P, \sigma \otimes \text{id})$ is Morita equivalent to an (L_P, R_P) -real division algebra with involution (D_P, ϑ_P) .

Let $S_P = \{Q \in \widetilde{X}_F \mid Q \text{ extends to } L_P\}$. By [16, Chap. 3, Thm. 4.4] we have $S_P = \{Q \in \widetilde{X}_F \mid \text{sign}_Q(\text{Tr}_{L_P/F}^*(1)) > 0\}$. The set S_P is therefore a clopen subset of \widetilde{X}_F containing P , and by compactness there are $P_1, \dots, P_t \in \widetilde{X}_F$ such that $\widetilde{X}_F = S_{P_1} \cup \dots \cup S_{P_t}$. It follows that there are disjoint clopen sets $S_i \subseteq S_{P_i}$ (for $i = 1, \dots, t$) such that $\widetilde{X}_F = S_1 \dot{\cup} \dots \dot{\cup} S_t$.

Let $i \in \{1, \dots, t\}$ and write $S := S_i$, $L := L_{P_i}$. It suffices to prove the lemma for S instead of \widetilde{X}_F . Let $Q \in S$ and let $R \in X_L/Q$. We have morphisms of Witt groups

$$W(A \otimes_F L, \sigma \otimes \text{id}) \xrightarrow{\mu} W(D, \vartheta) \xrightarrow{\rho} W(L),$$

where (D, ϑ) is (L, R) -real (note that $R \notin \text{Nil}[D, \vartheta]$ since $Q \notin \text{Nil}[A, \sigma]$ by Proposition 2.2), μ is an isomorphism induced by Morita equivalence and ρ is the map of Lemma 4.3. Let $H_0 = \langle 1 \rangle_{\vartheta} \in W(D, \vartheta)$ and let $\delta_R := \text{sgn}(\text{sign}_R^{\mu(H \otimes L)} H_0)$. Let $h \in W(A, \sigma)$ be arbitrary. Then, using Remark 4.2, we have

$$\begin{aligned} 4 \text{sign}_Q^H h &= 4 \text{sign}_R^{H \otimes L}(h \otimes L) \\ &= 4 \text{sign}_R^{\mu(H \otimes L)} \mu(h \otimes L) && [\text{by [1, Thm. 4.2]}] \\ &= 4 \text{sgn}(\text{sign}_R^{\mu(H \otimes L)} H_0) \text{sign}_R^{H_0} \mu(h \otimes L) \\ &= 4 \delta_R \text{sign}_R^{H_0} \mu(h \otimes L) \end{aligned}$$

$$\begin{aligned}
&= \delta_R \frac{4}{\ell} \text{sign}_R \rho(\mu(h \otimes L)) && \text{[by Lemma 4.3]} \\
&= \delta_R \text{sign}_R \varphi,
\end{aligned}$$

where $\ell = \dim_L D \in \{1, 2, 4\}$ and $\varphi := \frac{4}{\ell} \rho(\mu(h \otimes L)) \in W(L)$.

Observe that for every $R \in X_L/Q$, $\text{sign}_R^{\mu(H \otimes L)} H_0 \neq 0$ since H_0 is a reference form for (D, ϑ) . Since X_L/Q is finite, there is a finite tuple $\bar{a}_Q \in L$ of length ℓ_Q such that $H(\bar{a}_Q) \cap (X_L/Q)$ contains only one ordering R_Q . In particular

$$\sum_{R \in X_L/Q} \text{sign}_R^{\mu(H \otimes L)} (\langle\langle \bar{a}_Q \rangle\rangle \cdot H_0) = 2^{\ell_Q} \text{sign}_{R_Q}^{\mu(H \otimes L)} H_0 \neq 0.$$

Define $\varphi_Q := \langle\langle \bar{a}_Q \rangle\rangle \cdot \varphi$. Then, by [16, Chap. 3, Thm. 4.5],

$$\text{sign}_Q(\text{Tr}_{L/F}^* \varphi_Q) = \sum_{R \in X_L/Q} \text{sign}_R \varphi_Q = 2^{\ell_Q} \text{sign}_{R_Q} \varphi = 2^{\ell_Q+2} \delta_{R_Q} \text{sign}_Q^H h,$$

where $\text{Tr}_{L/F}^* : W(L) \rightarrow W(F)$ denotes the Scharlau transfer. Therefore the clopen subset of X_F

$$U_Q := (\text{sign}(\text{Tr}_{L/F}^* \varphi_Q) - 2^{\ell_Q+2} \delta_{R_Q} \text{sign}^H h)^{-1}(0)$$

contains Q , and thus $S = \bigcup_{Q \in X_F} U_Q$. Since S is compact we obtain $S = U_{Q_1} \cup \dots \cup U_{Q_r}$ for some $Q_1, \dots, Q_r \in S$, and for every $Q \in U_{Q_j}$:

$$2^{\ell_{Q_j}+2} \delta_{R_{Q_j}} \text{sign}_Q^H h = \text{sign}_Q(\text{Tr}_{L/F}^* \varphi_{Q_j}),$$

and thus

$$2^{\ell_{Q_j}+2} \text{sign}_Q^H h = \text{sign}_Q(\delta_{R_{Q_j}} \text{Tr}_{L/F}^* \varphi_{Q_j}).$$

Since $\delta_{R_{Q_j}} \text{Tr}_{L/F}^* \varphi_{Q_j} \in W(F)$, the result follows by taking clopen sets $U_j \subseteq U_{Q_j}$ such that $S = U_1 \dot{\cup} \dots \dot{\cup} U_r$. \square

Theorem 5.13. *There exists $n_0 \in \mathbb{N}$ such that for every $h \in W(A, \sigma)$ there exists $q_h \in W(F)$ with $2^{n_0} \text{sign}^H h = \text{sign} q_h$.*

Proof. We use the terminology of Lemma 5.12. Let $r := n_1 + \dots + n_t$. Then for every $h \in W(A, \sigma)$ and every $i \in \{1, \dots, t\}$ there exists $q_i \in W(F)$ such that $\text{sign} q_i = 2^r \text{sign}^H h$ on U_i .

Since t is finite, there exist $m \in \mathbb{N}$ and $p_i \in W(F)$ such that $\text{sign} p_i = 2^m$ on U_i and $\text{sign} p_i = 0$ on $X_F \setminus U_i$. Therefore $\text{sign}(p_i q_i) = 2^m \text{sign} q_i = 2^{m+r} \text{sign}^H h$ on U_i and $\text{sign}(p_i q_i) = 0$ on $X_F \setminus U_i$, and we obtain for $i = 1, \dots, t$,

$$\text{sign}(p_1 q_1 + \dots + p_t q_t) = \text{sign} p_i q_i \text{ on } U_i$$

$$= 2^{m+r} \text{sign}^H h \text{ on } U_i$$

and thus

$$\text{sign}(p_1 q_1 + \cdots + p_t q_t) = 2^{m+r} \text{sign}^H h \text{ on } X_F$$

(note that by construction the quadratic form $p_1 q_1 + \cdots + p_t q_t$ has zero signature on $\text{Nil}[A, \sigma]$). \square

Definition 5.14. We define

$$W_{\text{red}}(A, \sigma) := W(A, \sigma) / W_t(A, \sigma) \simeq \text{sign}^H(W(A, \sigma)) \subseteq \widetilde{C}(X_F, \mathbb{Z})$$

and call this the *reduced Witt group* of (A, σ) .

In order to compare reduced hermitian forms and reduced quadratic forms, we also introduce

$$\widetilde{W}_{\text{red}}(F) := \{q \in W_{\text{red}}(F) \mid \text{sign } q = 0 \text{ on } \text{Nil}[A, \sigma]\}.$$

Observe that $W_{\text{red}}(A, \sigma)$ is a $W_{\text{red}}(F)$ -module and also a $\widetilde{W}_{\text{red}}(F)$ -module in the natural way.

Proposition 5.15. *Let $h_0 \in W(A, \sigma)$ and $k_0 \in \mathbb{N}$ be such that $\text{sign}(h_0) = 2^{k_0}$ (cf. Lemma 5.2). With notation as in Theorem 5.13, the maps*

$$\widetilde{W}_{\text{red}}(F) \longrightarrow W_{\text{red}}(A, \sigma), \quad q \longmapsto q h_0$$

and

$$W_{\text{red}}(A, \sigma) \longrightarrow \widetilde{W}_{\text{red}}(F), \quad h \longmapsto q_h$$

are well-defined injective morphisms of $W_{\text{red}}(F)$ -modules.

Proof. Identifying $W_{\text{red}}(F)$ and $W_{\text{red}}(A, \sigma)$ with the images of sign and sign^H , we see that the first map is simply multiplication by 2^{k_0} .

The second map is well-defined because $h_1 = h_2$ in $W_{\text{red}}(A, \sigma)$ is equivalent to $\text{sign}^H h_1 = \text{sign}^H h_2$, which implies $\text{sign } q_{h_1} = \text{sign } q_{h_2}$, so $q_{h_1} = q_{h_2}$ in $\widetilde{W}_{\text{red}}(F)$. It is also easy to check that it is an injective morphism of $W_{\text{red}}(F)$ -modules. \square

We finish this paper by pointing out some difficulties that need to be overcome in order to further the study of the stability index of algebras with involution.

In the quadratic form literature one can find important links between the stability index of the field F and the powers of the fundamental ideal $I^n(F)$, which crucially depend on Pfister forms (see for example [4, Satz 3.17]). Although we can define $I(A, \sigma)$, the lack of a tensor product of hermitian forms in general is a serious obstacle to the development of analogous concepts and connections for Witt groups of algebras with involution.

Another issue is the following: the quadratic Pfister form $\langle 1, a \rangle$ has signature 2 on $H(a)$ and signature 0 on $H(-a)$, which is a fundamental observation when considering $\text{st}(F)$. In contrast, this behaviour cannot in general be replicated with hermitian forms since the H -signature of the form $\langle 1 \rangle_\sigma$ may not be constant and in addition may take values which are not in $\{-1, 1\}$.

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